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Symmetries of group–subgroup transformation factors for coupling, subduction and induction

R W Haase† and P H Butler‡

† Institut für Theoretische Physik, Technische Universität Wien, Karlsplatz 13, A-1040 Vienna, Austria

‡ Physics Department, University of Canterbury, Christchurch 1, New Zealand

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Abstract. Group-subgroup (GH) transformation theory uses G transformation coefficients and GH transformation factors to describe the relationship between the various bases of a representation space of any group decomposed into irreducible representation spaces of the group and its subgroups. The representation spaces may be formed by the processes of coupling, subduction or induction. We further develop this GH transformation theory with the study of the various symmetries (complex conjugation, transposition and associativity) of the GH transformation factors, emphasising the origins of each symmetry and a hierarchy within these symmetries. A method of calculation is given for certain transformation factors. The results and methods presented here generalise those of the Racah-Wigner coupling algebra.

1. Introduction

In a previous paper (Haase and Butler 1984a) we began a general introduction to what we have called the group-subgroup (GH) transformation theory. In that development several concepts of group theory and linear algebra were brought together. These included (i) reduction of representation spaces constructed by coupling, subduction and induction processes, (ii) the concept of G and GH bases and (iii) transformations between GH bases leading to definitions of a variety of transformation coefficients and factors.

Our development of this theory arose from consideration of the connection between the symmetric and unitary groups. We have called this relationship the Schur-Weyl duality. Many powerful equalities between certain transformation factors of the symmetric groups and those of the unitary groups can be established by this duality (see Haase and Butler (1984b) and references therein). One of the important aspects to arise is the occurrence of the duality factors which relate phase freedoms in the symmetric group with those in the unitary groups. To determine the matrix structure of these duality factors we need to know their symmetries.

With this aim in mind we continue the development of the GH transformation theory, concentrating on 'global' symmetries which may be found within quite general group-subgroup schemes but excluding those symmetries which pertain to specific groups. Our discussion parallels those of Derome (1966), Derome and Sharp (1965) and Butler (1975) for the Racah-Wigner coupling algebra, which is viewed as a special case of our theory.

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We begin with a general introduction of notation and terminology. This is presented in § 2 along with remarks on phase freedom within the GH transformation theory. Sections 3, 4 and 5 discuss respectively the symmetries of complex conjugation, transposition and associativity all with regard to coupled, subduced and induced representation spaces. We note a hierarchy within these symmetries. Specifically, the complex conjugation symmetry applies to all transformations within any representation space, whether just a single space or a direct product of two or more spaces. The transposition symmetry applies to direct products of two or more spaces, while a direct product of at least three spaces is required to display the associativity symmetry. For each symmetry we define a corresponding transformation factor which we call the complex conjugation factor, the transposition factor or the associativity factor. The symmetry hierarchy is emphasised by the fact that the transposition factor displays only complex conjugation symmetry, while the associativity factor displays both complex conjugation and transposition symmetries. We note that for any n-fold direct product group, the symmetries of the corresponding representation space can always be broken down to those of transposition and associativity. In particular this is shown for the fourfold product space from which is derived an important identity involving five associativity factors. This identity is used as a basis of a method for calculating associativity factors and parallels the use of the Biedenharn-Elliott sum rule in the calculation of $6j$ symbols (Butler and Wybourne 1976, Butler 1981). In § 5 we outline this procedure for the associativity factors.

In § 6 we discuss the symmetries of the coupling, subduction and induction factors which are those special transformation coefficients that we collectively call defining factors (defining because these factors describe the process by which irrep spaces are constructed from other irrep spaces). We shall also present new identities relating associativity factors and defining factors. These identities are analogous to the Wigner relation of the Racah-Wigner algebra relating $6j$ and $3jm$ symbols. Continuing the analogy in which the Wigner relation is used to calculate $3jm$ symbols (see Butler 1981), we propose a similar method for calculating the various defining factors.

2. Phase freedom

To begin we give some preliminary remarks on the GH transformation theory (see also Haase and Butler 1984a). We define a G basis of a representation space V of a compact continuous or finite group as a set of orthonormal basis vectors

$$\{|z\,\gamma(G)\,i\rangle \equiv |z\gamma i\rangle\colon i=1\ldots|\gamma|\} \tag{2.1}$$

labelled by the irreducible representations (irreps) $\gamma(G)$ of dimension $|\gamma|$. In addition the action of the group operators O_g is chosen to be independent of the parentage label z of G; the irrep matrices are identical for all the different equivalent irrep spaces

$$\langle z'\gamma' i'|O_g|z\gamma i\rangle = \delta^{z'}{}_{z}\delta^{\gamma'}{}_{\gamma}\gamma(g)^{i'}{}_{i}. \tag{2.2}$$

If two different G bases give an identical set of irrep matrices under the group action, they are termed equivalent G bases; if the two sets of irrep matrices are different, the G bases are said to be inequivalent.

Similar statements can be made of a GH basis. This basis is defined as a set of orthonormal basis vectors

$$\{|z\,\gamma(G)\,a\,\eta(H)\,j\rangle \equiv |z\gamma a\eta j\rangle\colon a=1\ldots|\gamma:\eta|,\ \eta(H), j=1\ldots|\eta|\} \tag{2.3}$$

that is simultaneously a G basis

$$\langle z'\gamma'a'\eta'j'|O_g|z\gamma a\eta j\rangle = \delta^{z'}{}_{z}\delta^{\gamma'}{}_{\gamma}\gamma(g)^{a'\eta'j'}{}_{a\eta j} \qquad \forall g\in G \tag{2.4}$$

and an H basis

$$\langle z'\gamma'a'\eta'j'|O_g|z\gamma a\eta j\rangle = \delta^{z'}{}_{z}\delta^{\gamma'}{}_{\gamma}\delta^{a'}{}_{z}\eta(g)^{j'}{}_{j} \qquad \forall g\in H. \tag{2.5}$$

Two GH bases are said to be equivalent if they give rise to identical sets of irrep matrices for both G and H; otherwise they are inequivalent GH bases.

We consider now a transformation between inequivalent GH bases but equivalent H bases. Such a transformation has the form

$$|z\gamma a\eta j\rangle = |z\gamma\hat{a}\eta j\rangle\langle\gamma\hat{a}\eta\,|\,\gamma a\eta\rangle. \tag{2.6}$$

The transformation factor $\langle\gamma\hat{a}\eta\,|\,\gamma a\eta\rangle$ is an element of a unitary matrix on the indices a and $\hat{a}$, and it can be seen to change some of the irrep matrices of $g\in G$

$$\gamma(g)^{a'\eta'j'}{}_{a\eta j} = \langle\gamma a'\eta'\,|\,\gamma\hat{a}'\eta'\rangle\gamma(g)^{\hat{a}'\eta'j'}{}_{\hat{a}\eta j}\langle\gamma\hat{a}\eta\,|\,\gamma a\eta\rangle \tag{2.7}$$

but leave invariant those of $g\in H$. These transformations are important in what we have termed the GH transformation theory which may be viewed as a generalisation of the Racah-Wigner coupling algebra. The transformation given by (2.6) specifies the freedom available in fixing the relative phase and multiplicity relationships between the basis vectors for a GH basis. This 'phase freedom' enables one to choose a GH basis which satisfies certain symmetry requirements that one wants to impose.

One can have general symmetries, such as the complex conjugation, permutation and associative symmetries, and symmetries arising from special properties of the group or groups in question, such as those symmetries generated by the one-dimensional irreps of a group (Butler and Ford 1979) and the Schur-Weyl duality symmetries of the unitary and symmetric groups (see Haase and Butler 1984b). We show how to give the matrices of the general symmetries their simplest possible form.

As a general notational point, note that a transformation factor in Dirac notation takes the archetypal form $\langle\gamma a'\eta\,|\,\gamma a\eta\rangle$ where (γ) and (η) are termed group (G) and subgroup (H) labels respectively and always appear twice, and (a') and (a) are GH basis labels appearing only once. (A transformation coefficient can be thought of as a GE transformation factor with E as the identity group.) Thus for a recoupling factor (see § 5)

$$\langle(\lambda\mu)a\eta,\,\nu,\,b\gamma\,|\,\lambda(\mu\nu)c\kappa,\,d\gamma\rangle$$

we identify $(\lambda\mu\nu)$ and (γ) as the G and H labels, and $(a\eta b)$ and $(c\kappa d)$ as the GH basis labels. A coupling factor (see § 6)

$$\langle\mu\nu a\gamma b\eta\,|\,(\mu c\lambda,\,\nu d\kappa)e\eta\rangle$$

has $(\mu\nu)$ and (η) as the G and H labels while $(a\gamma b)$ and $(c\lambda d\kappa e)$ form the GH basis labels respectively. Although we shall not be using the matrix form for these two factors, they could be written in the form

$$R(\lambda\mu\nu,\,\gamma)^{a\eta b}{}_{c\kappa d}, \qquad D(\mu\nu,\,\eta)^{a\gamma b}{}_{c\lambda d\kappa e}.$$

For simplicity we will use the matrix form to denote the transformation factors describing symmetries found within the GH transformation theory. Since phase freedoms play a key role in the choices of these transformation factors, the matrix form will also be used for them. Hence (2.6) is written

$$|z\gamma a\eta j\rangle = |z\gamma\hat{a}\eta j\rangle U(\gamma,\,\eta)^{\hat{a}}{}_{a}. \tag{2.8}$$

The summation convention used throughout this paper is then to sum on only *GH basis labels* (Greek or Latin) that occur once in a bra or raised in a matrix and once in a ket or lowered in a matrix.

In a similar manner as above, we may write down the phase freedom associated with the G basis $\{|y\eta\uparrow a\gamma i\rangle\}$ of the induced representation space $V_{y\eta\uparrow}$ (see Haase and Butler 1984a)

$$|y\eta\uparrow a\gamma i\rangle = |y\eta\uparrow \hat{a}\gamma i\rangle U(\eta\uparrow, \gamma)^{\hat{a}}{}_{a}. \tag{2.9}$$

For such a transformation the form of the irrep matrices of G remains the same. The similarity to (2.8) is due to the fact that the induced representation behaves in the reduced basis like any other representation of a group that has been decomposed into its constituent irreps. As we will see later this similarity recurs in the discussion of the symmetries of induced representation spaces.

Furthermore a similarity exists between the pair of archetypal transformation factors

$$\langle \gamma a'\eta \,|\, \gamma a\eta\rangle, \qquad \langle \eta\uparrow a'\gamma \,|\, \eta\uparrow a\gamma\rangle$$

which derives from the Frobenius reciprocity theorem. This states that the number of multiple occurrences of $\gamma(G)$ in $\eta(H)\uparrow G$ is equal to that of $\eta(H)$ in $\gamma(G)$. Since no equality holds between the two factors, the matrix form for one can be chosen for the other. This also applies for other transformation factors related by the Frobenius reciprocity—more specifically, the reciprocity relates subduction factors with induction factors and resubduction factors with reinduction factors (see §§ 5 and 6).

3. The complex conjugation symmetry

We use an operator approach similar to that given by Bickerstaff (1980) to obtain archetypal relations valid for any GH transformation coefficient. For a discussion of the complex conjugation symmetry and the reality criterion for coupling coefficients see Bickerstaff and Damhus (1984) and Bickerstaff (1985). The definition of Bickerstaff's complex conjugation operator k_γ differs from that of ours in that we have removed the representation space dependence into a linear operator $l_{\mathring{\gamma}}$, $k_\gamma = l_{\mathring{\gamma}}k$. The complex conjugation operator k is an antilinear, unitary, involutory operator mapping the representation space V onto its complex conjugate representation space denoted $\mathring{V}$. (Note $\mathring{V}$ may be equivalent to V.) That is, if $\alpha \in \mathbb{C}$, $|u\rangle, |v\rangle \in V$, $|\mathring{u}\rangle \in \mathring{V}$ then

$$kk = e \qquad \text{where } e \text{ is the identity operator,} \tag{3.1}$$

$$k\cdot|u\rangle\alpha = |\mathring{u}\rangle\alpha^*, \tag{3.2}$$

$$\langle k\cdot u\,|\,k\cdot v\rangle = \langle \mathring{u}\,|\,\mathring{v}\rangle = \langle u\,|\,v\rangle^*. \tag{3.3}$$

We shall also require that k commutes with all the group operators

$$kO_g = O_g k \qquad \forall g \in G. \tag{3.4}$$

With the above criteria we consider the action of k on G bases of $V_{z\gamma}$. This action gives a unique basis, called the complex conjugate basis, for $V_{\mathring{z}\mathring{\gamma}}$, which is a G basis but not necessarily equivalent to the G basis already chosen for $V_{\mathring{z}\mathring{\gamma}}$. Thus we have

$$k\cdot|z\gamma i\rangle \equiv |\mathring{z}\mathring{\gamma}\mathring{i}\rangle \tag{3.5}$$

with the commutation requirement giving the group action on the basis $\{|\overset{*}{z}\overset{*}{\gamma}\overset{*}{i}\rangle\}$ as

$$\overset{*}{\gamma}(g)^{\overset{*}{j}}{}_{\overset{*}{i}} = \gamma(g)^{j}{}_{i}{}^{*}. \tag{3.6}$$

This gives the 'matrix elements' of k as

$$\langle \overset{*}{z}\overset{*}{\gamma}\overset{*}{j} \,|\, k\cdot z\gamma i\rangle \equiv K(\gamma)^{ji} = \delta^{ji}. \tag{3.7}$$

The antilinearity of k implies that

$$\langle z\gamma i \,|\, k^{\dagger}\cdot \overset{*}{z}\overset{*}{\gamma}\overset{*}{j}\rangle \equiv K(\overset{*}{\gamma})^{\dagger ij} = K(\gamma)^{ji*}. \tag{3.8}$$

We have raised the index i in the definition of the matrix $K(\gamma)$ because, like the index j, the scalar product $\langle \overset{*}{z}\overset{*}{\gamma}\overset{*}{j} \,|\, k\cdot z\gamma i\rangle$ is antilinear in i. The matrix $K(\gamma)$ is also independent of the parentage label since we only consider G bases. The involutary nature of k implies that the basis $\{|\overset{**}{z}\,\overset{**}{\gamma}\,\overset{**}{i}\rangle\}$ given by

$$kk\cdot|z\gamma i\rangle \equiv |\overset{**}{z}\,\overset{**}{\gamma}\,\overset{**}{i}\rangle \tag{3.9}$$

is the same G basis as $\{|z\gamma i\rangle\}$

$$\overset{**}{\gamma}(g)^{\overset{**}{j}}{}_{\overset{**}{i}} = \gamma(g)^{j}{}_{i}. \tag{3.10}$$

Similar statements can be made for GH bases $\{|z\gamma a\eta j\rangle\}$ by replacing (i) by $(a\eta j)$ and also for G bases of $V_{y\eta\uparrow}$, $\{|y\eta\uparrow a\gamma i\rangle\}$ by replacing (z) by $(y\eta\uparrow a)$.

As stated above, the complex conjugate basis is not necessarily the chosen G basis for $V_{\overset{*}{z}\overset{*}{\gamma}}$. A transformation is required, for example

$$|\overset{*}{z}\overset{*}{\gamma}\overset{*}{i}\rangle = |\overset{*}{z}\overset{*}{\gamma}i'\rangle A(\overset{*}{\gamma})^{i'}{}_{\overset{*}{i}} \tag{3.11}$$

or if $\{|z\gamma\overset{*}{i}'\rangle\}$ is the complex conjugate basis formed from the G basis $\{|\overset{*}{z}\overset{*}{\gamma}i'\rangle\}$

$$|z\gamma\overset{*}{i}'\rangle = |z\gamma i\rangle A(\gamma)^{i}{}_{\overset{*}{i}'}. \tag{3.12}$$

Bickerstaff (1980) calls $A(\overset{*}{\gamma})$ a complex conjugation matrix and its elements complex conjugation coefficients. We call them simply the A matrix and A coefficients (after Derome and Sharp 1965). They have the unitary property

$$A(\overset{*}{\gamma})^{i'}{}_{\overset{*}{i}}A(\overset{*}{\gamma})^{\dagger\overset{*}{i}}{}_{j'} = \delta^{i'}{}_{j'}, \qquad A(\overset{*}{\gamma})^{\dagger\overset{*}{i}}{}_{i'}A(\overset{*}{\gamma})^{i'}{}_{\overset{*}{j}} = \delta^{j}{}_{i}. \tag{3.13}$$

We associate the linear operator $l_{\overset{*}{\gamma}}$ with each A matrix. It has matrix elements

$$\langle \overset{*}{z}\overset{*}{\gamma}i'|l_{\overset{*}{\gamma}}|\overset{*}{z}\overset{*}{\gamma}\overset{*}{i}\rangle = A(\overset{*}{\gamma})^{i'}{}_{\overset{*}{i}}. \tag{3.14}$$

The operator $(l_{\gamma}kl_{\overset{*}{\gamma}}k)$ is a linear operator mapping the G basis of V_{γ} into an equivalent G basis. By Schur's lemmas we then have

$$(l_{\gamma}kl_{\overset{*}{\gamma}}k) = \{\gamma\}e \tag{3.15}$$

where $\{\gamma\}$ is called a $1j$ phase. A simple manipulation starting from $(l_{\overset{*}{\gamma}}kl_{\gamma}k) = (l_{\overset{*}{\gamma}}k)(l_{\gamma}kl_{\overset{*}{\gamma}}k)(l_{\overset{*}{\gamma}}k)^{\dagger}$ implies $\{\overset{*}{\gamma}\} = \{\gamma\}^{*}$. In terms of the A coefficients (3.15) gives

$$A(\gamma)^{i}{}_{\overset{*}{i}'}A(\overset{*}{\gamma})^{i'}{}_{\overset{*}{i}}{}^{*} = \{\gamma\}\delta^{i}{}_{j}. \tag{3.16}$$

The A coefficients are related to the $2jm$ symbol of the Racah–Wigner coupling algebra (see Butler 1981)

$$\begin{pmatrix} \overset{*}{\gamma} & \gamma \\ i' & i \end{pmatrix}^{*} \equiv A(\gamma)^{i'i} = A(\overset{*}{\gamma})^{i'}{}_{\overset{*}{j}}K(\gamma)^{ji}. \tag{3.17}$$

However, we use the above notation and terminology since we want to establish a more general setting for the complex conjugation symmetry within the GH transformation. One result is the following.

Given any transformation between alternative G bases of $V_{z\gamma}$, say $\langle \gamma i | \gamma j \rangle$, the complex conjugation operator induces a similar transformation between alternative G bases of $V_{\overset{*}{z}\overset{*}{\gamma}}$

$$\langle \overset{*}{\gamma} i' | \overset{*}{\gamma} j' \rangle = A(\overset{*}{\gamma})^{i'}{}_{\overset{*}{i}} \langle \gamma i | \gamma j \rangle^* A(\overset{*}{\gamma})^{\dagger \overset{*}{j}}{}_{j'}. \tag{3.18}$$

This relation is the archetypal relation of the complex conjugation symmetry for any transformation coefficient. It is applicable not only to single irrep spaces $V_{z\gamma}$ but also to direct product irrep spaces $V_{\gamma_1} \otimes V_{\gamma_2} \simeq V_{\gamma_1 \otimes \gamma_2}$. In particular the complex conjugation symmetry of the coupling coefficients, called the Derome–Sharp lemma in the literature, can be derived from (3.18) (see Bickerstaff 1980). However, to show this one needs to consider the complex conjugation symmetry with respect to a GH basis.

The action of k on a GH basis can be defined as

$$k \cdot |z\gamma a \eta j\rangle \equiv |\overset{*}{z}\overset{*}{\gamma}\overset{*}{a}\overset{*}{\eta}\overset{*}{j}\rangle. \tag{3.19}$$

We use (3.12) to transform this complex conjugate basis to the chosen H basis

$$|\overset{*}{z}\overset{*}{\gamma}\overset{*}{a}\overset{*}{\eta}\overset{*}{j}\rangle = |\overset{*}{z}\overset{*}{\gamma}\overset{*}{a}\overset{*}{\eta} j'\rangle A(\overset{*}{\eta})^{j'}{}_{\overset{*}{j}}. \tag{3.20}$$

Still the GH basis $\{|\overset{*}{z}\overset{*}{\gamma}\overset{*}{a}\overset{*}{\eta} j'\rangle\}$ may not be the chosen GH basis for $V_{\overset{*}{z}\overset{*}{\gamma}}$ and a further transformation may be required. This introduces the complex conjugation factor

$$|\overset{*}{z}\overset{*}{\gamma}\overset{*}{a}\overset{*}{\eta}\overset{*}{j}\rangle = |\overset{*}{z}\overset{*}{\gamma} a' \overset{*}{\eta} j'\rangle A(\overset{*}{\gamma}, \overset{*}{\eta})^{a'}{}_{\overset{*}{a}} A(\overset{*}{\eta})^{j'}{}_{\overset{*}{j}} \tag{3.21}$$

(cf (2.8)). The factorisation of (3.21) into two transformations is the result of the fact that the GH basis is simultaneously an H basis. From the properties of k the A factors satisfy the following:

$$A(\overset{*}{\gamma}, \overset{*}{\eta})^{\dagger a}{}_{\overset{*}{a}'} A(\overset{*}{\gamma}, \overset{*}{\eta})^{a'}{}_{\overset{*}{b}} = \delta^b{}_a, \tag{3.22}$$

$$A(\overset{*}{\gamma}, \overset{*}{\eta})^{a'}{}_{\overset{*}{a}} A(\overset{*}{\gamma}, \overset{*}{\eta})^{\dagger a}{}_{\overset{*}{b}'} = \delta^{a'}{}_{b'}, \tag{3.23}$$

$$A(\gamma, \eta)^a{}_{\overset{*}{a}'} A(\overset{*}{\gamma}, \overset{*}{\eta})^{a'}{}_{\overset{*}{b}}{}^* = \{\gamma\}\{\eta\}^* \delta^a{}_b. \tag{3.24}$$

Applying the complex conjugation operator to inequivalent GH bases but equivalent H bases gives

$$\langle \overset{*}{\gamma} b' \overset{*}{\eta} | \overset{*}{\gamma} a' \overset{*}{\eta} \rangle = A(\overset{*}{\gamma}, \overset{*}{\eta})^{b'}{}_{\overset{*}{b}} \langle \gamma b \eta | \gamma a \eta \rangle^* A(\overset{*}{\gamma}, \overset{*}{\eta})^{\dagger \overset{*}{a}}{}_{a'}. \tag{3.25}$$

This relation describes the complex conjugation symmetry for any GH transformation factor, in particular for the coupling factor and the subduction factor. Next we consider the complex conjugation for induced representations. This aspect has not been previously discussed in the literature. The results are similar to the above.

The action of k on the G basis of an induced representation space is defined in an analogous way to the action of a GH basis:

$$k \cdot |y\eta\uparrow a\gamma i\rangle \equiv |\overset{*}{y}\overset{*}{\eta}\uparrow \overset{*}{a}\overset{*}{\gamma}\overset{*}{i}\rangle = |\overset{*}{y}\overset{*}{\eta}\uparrow a' \overset{*}{\gamma} i'\rangle A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma})^{a'}{}_{\overset{*}{a}} A(\overset{*}{\gamma})^{i'}{}_{\overset{*}{i}} \tag{3.26}$$

(cf (3.22)). The A factor $A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma})$ transforms between the intermediate basis $\{|\overset{*}{y}\overset{*}{\eta}\uparrow \overset{*}{a}\overset{*}{\gamma} i'\rangle\}$ and the chosen G basis $\{|\overset{*}{y}\overset{*}{\eta}\uparrow a' \overset{*}{\gamma} i'\rangle\}$. These A factors can be shown to have the properties

$$A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma})^{\dagger \overset{*}{a}}{}_{a'} A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma})^{a'}{}_{\overset{*}{b}} = \delta^b{}_a, \tag{3.27}$$

$$A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma})^{a'}{}_{\overset{*}{a}} A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma})^{\dagger \overset{*}{a}}{}_{b'} = \delta^{a'}{}_{b'}, \tag{3.28}$$

$$A(\eta\uparrow, \gamma)^{a}{}_{\overset{*}{a}'} A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma})^{a'}{}_{\overset{*}{b}}{}^{*} = \{\eta\}\{\gamma\}^{*} = \delta^{a}{}_{b}. \tag{3.29}$$

The last result follows from the fact that k commutes with the group operators (in particular the coset operators of G/H) and from the irreducibility of $\eta(H)$. Furthermore if the induction factor $\langle \eta\uparrow b\gamma | \eta\uparrow a\gamma \rangle$ describes the transformation between two alternative G bases for $V_{y\eta\uparrow}$ then the complex conjugation operator induces a transformation between G bases of $V_{\overset{*}{y}\overset{*}{\eta}\uparrow}$

$$\langle \overset{*}{\eta}\uparrow b'\overset{*}{\gamma} | \overset{*}{\eta}\uparrow a'\overset{*}{\gamma} \rangle = A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma})^{b'}{}_{\overset{*}{b}} \langle \eta\uparrow b\gamma | \eta\uparrow a\gamma \rangle^{*} A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma})^{\dagger \overset{*}{a}}{}_{a'}. \tag{3.30}$$

This relation describes the complex conjugation symmetry for any induction factor.

Having defined the complex conjugation factors $A(\overset{*}{\gamma}, \overset{*}{\eta})^{a'}{}_{a}$ and $A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma})^{a'}{}_{a}$ we now determine their matrix structure. The possible matrix choices are given by the phase freedom matrices of (2.8) and (2.9). In matrix form we have

$$\hat{A}(\overset{*}{\gamma}, \overset{*}{\eta}) = U(\overset{*}{\gamma}, \overset{*}{\eta}) A(\overset{*}{\gamma}, \overset{*}{\eta}) U(\gamma, \eta)^{\mathrm{T}}, \tag{3.31}$$

$$\hat{A}(\overset{*}{\eta}\uparrow, \overset{*}{\gamma}) = U(\overset{*}{\eta}\uparrow, \overset{*}{\gamma}) A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma}) U(\eta\uparrow, \gamma)^{\mathrm{T}}. \tag{3.32}$$

Since the complex conjugation symmetry is a symmetry which can be imposed for any and every type of transformation factor we assume that complete phase freedom exists and that all the various types of complex conjugation factors will take the same matrix form as the archetypal $A(\gamma, \eta)$ and $A(\eta\uparrow, \gamma)$ which we now give. Bickerstaff (1980) has discussed the choices for the former. The argument extends simply to the latter. Two cases arise.

(1) If γ or η is a complex irrep we can always choose

$$\hat{A}(\gamma, \eta) = \mathbb{1} \qquad \text{and} \qquad \hat{A}(\eta\uparrow, \gamma) = \mathbb{1} \tag{3.33}$$

since in (3.31) and (3.32) $U(\gamma, \eta)$ and $U(\overset{*}{\gamma}, \overset{*}{\eta})$ (respectively $U(\eta\uparrow, \gamma)$ and $U(\overset{*}{\eta}\uparrow, \overset{*}{\gamma})$) are distinct.

(2) If γ and η are both real then by (3.24) and (3.29) we have two possibilities.

(a) If $\{\gamma\}\{\eta\}^{*} = +1$ then $A(\gamma, \eta)$ and $A(\eta\uparrow, \gamma)$ are symmetric matrices. However, we can still make the choice

$$\hat{A}(\gamma, \eta) = \mathbb{1} \qquad \text{and} \qquad \hat{A}(\eta\uparrow, \gamma) = \mathbb{1}. \tag{3.34}$$

(b) If $\{\gamma\}\{\eta\}^{*} = -1$ then $A(\gamma, \eta)$ and $A(\eta\uparrow, \gamma)$ are skew-symmetric matrices and may be chosen

$$\hat{A}(\gamma, \eta) = \mathbb{J} \qquad \text{and} \qquad \hat{A}(\eta\uparrow, \gamma) = \mathbb{J} \tag{3.35}$$

where $\mathbb{J} = \left(\begin{smallmatrix} 0 & +1 \\ -1 & 0 \end{smallmatrix}\right) \otimes \mathbb{1}$.

In requiring that any other GH transformations do not change the above choices of the A factors, we must constrain the phase freedom matrices by

$$U(\overset{*}{\gamma}, \overset{*}{\eta}) = A(\overset{*}{\gamma}, \overset{*}{\eta}) U(\gamma, \eta)^{*} A(\overset{*}{\gamma}, \overset{*}{\eta})^{\dagger} \tag{3.36}$$

$$U(\overset{*}{\eta}\uparrow, \overset{*}{\gamma}) = A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma}) U(\eta\uparrow, \gamma)^{*} A(\overset{*}{\eta}\uparrow, \overset{*}{\gamma})^{\dagger}. \tag{3.37}$$

In particular for case (3.34), $U(\cdot,\cdot)$ must be real and hence orthogonal, while for case (3.35) we have a symplectic $U(\cdot,\cdot)$.

Although the above choices would appear to be the simplest it must be remembered that other choices do exist and may be more desirable; for example, other choices of transformation factors may turn out to be real. To illustrate this point we refer the reader to the table of SU_3 $6j$ symbols of Bickerstaff *et al* (1982). If their unity choice of the A matrices for the coupling $\{21\}\times\{21\}\supset\{21\}$ is replaced by

$$A(21\ \ 21, 21)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{3.38}$$

the $6j$ symbols containing this coupling take real values (Sullivan 1983).

4. The transposition symmetry

The appearance of a direct product group, $H\times K$ say, in a group-subgroup (GH) chain leads to a further symmetry—that of transposing the two groups H and K. When the two groups are the same the symmetry is non-trivial. In this section we give a detailed discussion of the transposition symmetry with respect to coupled, subduced and induced representation spaces. The three types of spaces reflecting the transposition symmetry are labelled by the bases

$$\begin{aligned}
&\text{(A) of coupling} && \{|xy\ \eta\kappa(H\times K)\ a\ \gamma(G)\ i\rangle\},\\
&\text{(B) of subduction} && \{|z\ \gamma(G)\ a\ \eta\kappa(H\times K)\ jk\rangle\},\\
&\text{(C) of induction} && \{|xy\ \eta\kappa(H\times K)\!\uparrow\! G\ a\ \gamma(G)\ i\rangle\},
\end{aligned} \tag{4.1}$$

where x, y, z are parentage labels of η, κ, γ respectively.

We define the transposition operator τ as a linear unitary, involutary operator mapping the direct product space $V_1\otimes V_2$ into $V_2\otimes V_1$; that is, $\alpha\in\mathbb{C}$; $|u_1\rangle$, $|u_1'\rangle\in V_1$; $|u_2\rangle$, $|u_2'\rangle\in V_2$;

$$\tau\tau=e, \tag{4.2}$$

$$\tau\cdot|u_1u_2\rangle\alpha=|u_2u_1\rangle\alpha, \tag{4.3}$$

$$\langle\tau\cdot u_1'u_2'|\tau\cdot u_1u_2\rangle=\langle u_2'u_1'|u_2u_1\rangle. \tag{4.4}$$

The action of τ on the basis vectors (4.1) defines then what we have called the transposition factors or more simply the T factors. Like the phase freedom factors and the A factors, they describe the transformation from the GH basis labelled in part by $\eta\kappa(H\times K)$ to the GH basis labelled by $\kappa\eta(K\times H)$:

$$\begin{aligned}
&\text{(A)} && \tau\cdot|xy\eta\kappa a\gamma i\rangle=|yx\kappa\eta a'\gamma i\rangle T(\eta\kappa,\gamma)^{a'}{}_{a},\\
&\text{(B)} && \tau\cdot|z\gamma a\eta\kappa jk\rangle=|z\gamma a'\kappa\eta kj\rangle T(\gamma,\eta\kappa)^{a'}{}_{a},\\
&\text{(C)} && \tau\cdot|xy\eta\kappa\!\uparrow\! a\gamma i\rangle\mathrel{\overline{\overline{\tau}}}|yx\kappa\eta\!\uparrow\! a'\gamma i\rangle T(\eta\kappa\!\uparrow,\gamma)^{a'}{}_{a}.
\end{aligned} \tag{4.5}$$

Each of these T factors satisfies unitary conditions with summation over indices a or a'. The involutary property gives an additional symmetry relation. In matrix form we have

$$\begin{aligned}
&\text{(A)} && T(\kappa\eta,\gamma)T(\eta\kappa,\gamma)=\mathbb{1},\\
&\text{(B)} && T(\gamma,\kappa\eta)T(\gamma,\eta\kappa)=\mathbb{1},\\
&\text{(C)} && T(\kappa\eta\!\uparrow,\gamma)T(\eta\kappa\!\uparrow,\gamma)=\mathbb{1}.
\end{aligned} \tag{4.6}$$

For the special case when $\kappa(K)=\eta(H)$ the T matrices must be Hermitian. This will be important when we come to making choices of the T factors. The complex conjugation symmetry follows immediately from (3.25) and (3.30) by replacing the labels $\gamma(G)$ and $\eta(H)$ with the appropriate $\eta\kappa(H\times K)$ and $\gamma(G)$ labels. Thus we have

$$\begin{aligned}
&\text{(A)} \quad T(\mathring{\eta}\mathring{\kappa}, \mathring{\gamma}) = A(\mathring{\kappa}\mathring{\eta}, \mathring{\gamma})T(\eta\kappa, \gamma)^* A(\mathring{\eta}\mathring{\kappa}, \mathring{\gamma})^\dagger,\\
&\text{(B)} \quad T(\mathring{\gamma}, \mathring{\eta}\mathring{\kappa}) = A(\mathring{\gamma}, \mathring{\kappa}\mathring{\eta})T(\gamma, \eta\kappa)^* A(\mathring{\gamma}, \mathring{\eta}\mathring{\kappa})^\dagger,\\
&\text{(C)} \quad T(\mathring{\eta}\mathring{\kappa}\uparrow, \mathring{\gamma}) = A(\mathring{\kappa}\mathring{\eta}\uparrow, \mathring{\gamma})T(\eta\kappa\uparrow, \gamma)^* A(\mathring{\eta}\mathring{\kappa}\uparrow, \mathring{\gamma})^\dagger.
\end{aligned} \tag{4.7}$$

Special attention must be given to two cases.

(1) If $H=K$, $\kappa=\mathring{\eta}\neq\eta$ and $\mathring{\gamma}=\gamma$, we can combine (3.24) and (3.29) with (4.7) to give

$$\begin{aligned}
&\text{(A)} \quad T(\mathring{\eta}\eta, \gamma)A(\mathring{\eta}\eta, \gamma) = \{\gamma\}[T(\mathring{\eta}\eta, \gamma)A(\mathring{\eta}\eta, \gamma)]^{\mathrm{T}},\\
&\text{(B)} \quad T(\gamma, \mathring{\eta}\eta)A(\gamma, \mathring{\eta}\eta) = \{\gamma\}[T(\gamma, \mathring{\eta}\eta)A(\gamma, \mathring{\eta}\eta)]^{\mathrm{T}},\\
&\text{(C)} \quad T(\mathring{\eta}\eta\uparrow, \gamma)A(\mathring{\eta}\eta\uparrow, \gamma) = \{\gamma\}[T(\mathring{\eta}\eta\uparrow, \gamma)A(\mathring{\eta}\eta\uparrow, \gamma)]^{\mathrm{T}}.
\end{aligned} \tag{4.8}$$

That is, the product $T(\ldots)A(\ldots)$ is symmetric if γ is orthogonal and skew-symmetric if γ is symplectic.

(2) If all the irreps are real, the T matrix satisfies

$$T(\ldots)A(\ldots) = A(\ldots)T(\ldots). \tag{4.9}$$

For all other cases the symmetries given by (4.6)-(4.7) relate distinct T matrices.

To determine the possible matrix choices for the T matrices the phase freedom must be established:

$$\begin{aligned}
&\text{(A)} \quad \hat{T}(\eta\kappa, \gamma) = U(\kappa\eta, \gamma)T(\eta\kappa, \gamma)U(\eta\kappa, \gamma)^\dagger,\\
&\text{(B)} \quad \hat{T}(\gamma, \eta\kappa) = U(\gamma, \kappa\eta)T(\gamma, \eta\kappa)U(\gamma, \eta\kappa)^\dagger,\\
&\text{(C)} \quad \hat{T}(\eta\kappa\uparrow, \gamma) = U(\kappa\eta\uparrow, \gamma)T(\eta\kappa\uparrow, \gamma)U(\eta\kappa\uparrow, \gamma)^\dagger.
\end{aligned} \tag{4.10}$$

Each $U(\ldots)$ is subject to constraints (3.36)-(3.37). However, there is still sufficient freedom to choose all T matrices diagonal:

$$\begin{aligned}
&\text{(A)} \quad \hat{T}(\eta\kappa, \gamma)^{a'}{}_{a} = \{\eta\kappa a\gamma\}\delta^{a'}{}_{a},\\
&\text{(B)} \quad \hat{T}(\gamma, \eta\kappa)^{a'}{}_{a} = \{\gamma a\eta\kappa\}\delta^{a'}{}_{a},\\
&\text{(C)} \quad \hat{T}(\eta\kappa\uparrow, \gamma)^{a'}{}_{a} = \{\eta\kappa\uparrow a\gamma\}\delta^{a'}{}_{a},
\end{aligned} \tag{4.11}$$

where $\{\eta\kappa a\gamma\}$, $\{\gamma a\eta\kappa\}$ and $\{\eta\kappa\uparrow a\gamma\}$ are phases which we shall call transposition (or T) phases and are arbitrary except when:

(a) $\eta=\kappa$ and $H=K$. By (4.6), $T(\ldots)$ is Hermitian and

$$\{\eta\eta a\gamma\} = \{\gamma a\eta\eta\} = \{\eta\eta\uparrow a\gamma\} = \pm 1. \tag{4.12}$$

The T phases are fixed by the character theory of symmetrised Kronecker squares.

(b) $\kappa=\mathring{\eta}\neq\eta$ and $H=K$. Then

(i) if $\{\gamma\}=+1$, the T phases are arbitrary for all a;

(ii) if $\{\gamma\}=-1$, the T phases corresponding to a T matrix are fixed in pairs but are otherwise arbitrary. The multiplicity in these cases is always even.

The form of the T matrices of (a) and (b) can always be chosen as

$$\begin{pmatrix} \mathbb{1}_s & 0 \\ 0 & -\mathbb{1}_a \end{pmatrix}, \qquad \mathbb{D}, \qquad \begin{pmatrix} \mathbb{D}' & 0 \\ 0 & \mathbb{D}' \end{pmatrix}, \tag{4.13}$$

with $\mathbb{1}_s$, $\mathbb{1}_a$ as unit matrices and $\mathbb{D}$ and $\mathbb{D}'$ diagonal matrices.

For the choices to remain invariant under further GH transformations the phase freedom matrices must be further constrained:

$$\begin{aligned} &(\mathrm{A}) \qquad U(\kappa\eta, \gamma) = T(\eta\kappa, \gamma) U(\eta\kappa, \gamma) T(\eta\kappa, \gamma)^\dagger, \\ &(\mathrm{B}) \qquad U(\gamma, \kappa\eta) = T(\gamma, \eta\kappa) U(\gamma, \eta\kappa) T(\gamma, \eta\kappa)^\dagger, \\ &(\mathrm{C}) \qquad U(\kappa\eta\uparrow, \gamma) = T(\eta\kappa\uparrow, \gamma) U(\eta\kappa\uparrow, \gamma) T(\eta\kappa\uparrow, \gamma)^\dagger. \end{aligned} \tag{4.14}$$

In the case $\eta(H) = \kappa(K)$ the restrictions imply that the corresponding phase freedom is block diagonal,

$$T(\ldots) = \begin{pmatrix} T_s & 0 \\ 0 & T_a \end{pmatrix}. \tag{4.15}$$

Given any transformation factor referring to bases containing a direct product of two or more groups, the transposition operator can be used to induce a transformation in the 'transposed' space. We can write the archetypal equations as

$$\begin{aligned} &(\mathrm{A}) \qquad \langle \kappa\eta b'\gamma | \kappa\eta a'\gamma \rangle = T(\eta\kappa, \gamma)^{b'}{}_b \langle \eta\kappa b\gamma | \eta\kappa a\gamma \rangle T(\eta\kappa, \gamma)^{\dagger a}{}_{a'}, \\ &(\mathrm{B}) \qquad \langle \gamma b'\kappa\eta | \gamma a'\kappa\eta \rangle = T(\gamma, \eta\kappa)^{b'}{}_b \langle \gamma b\eta\kappa | \gamma a\eta\kappa \rangle T(\gamma, \eta\kappa)^{\dagger a}{}_{a'}, \\ &(\mathrm{C}) \qquad \langle \kappa\eta\uparrow b'\gamma | \kappa\eta\uparrow a'\eta \rangle = T(\eta\kappa\uparrow, \gamma)^{b'}{}_b \langle \eta\kappa\uparrow b\gamma | \eta\kappa\uparrow a\gamma \rangle T(\eta\kappa\uparrow, \gamma)^{\dagger a}{}_{a'}. \end{aligned} \tag{4.16}$$

Haase and Butler (1985) and Haase and Dirl (1985) exploited this transposition symmetry to obtain relations between $3jm$ symbols of $\mathrm{U}_{mn} \supset \mathrm{U}_m \times \mathrm{U}_n$ and those of $\mathrm{U}_{mn} \supset \mathrm{U}_n \times \mathrm{U}_m$ and between $3jm$ symbols of $\mathrm{S}_{f_1+f_2} \supset \mathrm{S}_{f_1} \times \mathrm{S}_{f_2}$ and those of $\mathrm{S}_{f_2+f_1} \supset \mathrm{S}_{f_2} \times \mathrm{S}_{f_1}$. We have

$$\begin{aligned} &\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ a_1 & a_2 & a_3 \\ \eta_1\kappa_1 & \eta_2\kappa_2 & \eta_3\kappa_3 \end{pmatrix}^{r}_{st} \\ &\quad = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ a_1' & a_2' & a_3' \\ \kappa_1\eta_1 & \kappa_2\eta_2 & \kappa_3\eta_3 \end{pmatrix}^{r'}_{ts} T(\gamma_1, \eta_1\kappa_1)^{a_1'}{}_{a_1} T(\gamma_2, \eta_2\kappa_2)^{a_2'}{}_{a_2} T(\gamma_3, \eta_3\kappa_3)^{a_3'}{}_{a_3}. \end{aligned} \tag{4.17}$$

(Note the independence of the symmetry with respect to the coupled product multiplicity labels r, s, t.) In the algebraic formula of some $3jm$ symbols the factors $(m-n)$ for U_{mn} and $(f_1 - f_2)$ for $\mathrm{S}_{f_1+f_2}$ may appear. The transposition symmetry explains the null values of these $3jm$ symbols when $m = n$ and $f_1 = f_2$ respectively.

5. The associativity symmetry

A further symmetry arises when there occurs a direct product of three or more groups L, M, N, ... There is then more than one way of forming pairwise the direct product of these groups. Indeed if there are n groups the number of ways of forming the n-fold direct product, without including permutations of the groups, is given by the recursive formula

$$a(n)=\sum_{1}^{n-1} a(m)a(n-m) \qquad \text{with } a(0)=a(1)=1.$$

The first few terms, $n=2, 3, 4, 5, 6$, take the values $a(n)=1, 2, 5, 14, 42$ respectively. In this section we will only be considering the direct product of three and four groups. In general the n-fold direct product can always be reduced to the consideration of threefold products as we will show for the fourfold direct product. This last case will lead to important identities.

There are two ways of pairing the direct product of three groups L, M, N given a particular order. These are denoted $((L\times M)\times N))$ and $(L\times(M\times N))$. Thus, given the irreps $\lambda(L)$, $\mu(M)$, $\nu(N)$ and a process by which they are paired (that is, either coupling, subduction or induction) the associativity symmetry then gives two different GH chains for labelling the basis vectors. The scalar product between these two basis vectors defines the recoupling, resubduction and reinduction factors (see Haase and Butler 1984a):

$$\begin{aligned}
&\text{(A)} \qquad \langle(\lambda\mu)a\eta,\ \nu,\ b\gamma\,|\,\lambda(\mu\nu)c\kappa,\ d\gamma\rangle,\\
&\text{(B)} \qquad \langle\gamma a\eta(b\lambda\mu)\nu\,|\,\gamma c\lambda\kappa(d\mu\nu)\rangle, \qquad\qquad (5.1)\\
&\text{(C)} \qquad \langle(\lambda\mu)\!\uparrow a\eta,\ \nu,\ \uparrow b\gamma\,|\,\lambda(\mu\nu)\!\uparrow c\kappa,\ \uparrow d\gamma\rangle.
\end{aligned}$$

We will collectively call these three factors associativity factors—the term being taken from algebra. They have similar properties. The complex conjugation symmetry is obtained from (3.25) and (3.30):

$$\begin{aligned}
&\text{(A)}\ \langle(\overset{*}{\lambda}\overset{*}{\mu})a'\overset{*}{\eta},\ \overset{*}{\nu},\ b'\overset{*}{\gamma}\,|\,\overset{*}{\lambda}(\overset{*}{\mu}\overset{*}{\nu})c'\overset{*}{\kappa},\ d'\overset{*}{\gamma}\rangle\\
&\qquad =A(\overset{*}{\lambda}\overset{*}{\mu},\ \overset{*}{\eta})^{a'}{}_{\overset{*}{a}}A(\overset{*}{\eta}\overset{*}{\nu},\ \overset{*}{\gamma})^{b'}{}_{\overset{*}{b}}\langle(\lambda\mu)a\eta,\ \nu,\ b\gamma\,|\,\lambda(\mu\nu)c\kappa,\ d\gamma\rangle^{*}\\
&\qquad\quad \times A(\overset{*}{\mu}\overset{*}{\nu},\ \overset{*}{\kappa})^{\dagger\overset{*}{c}}{}_{c'}A(\overset{*}{\lambda}\overset{*}{\kappa},\ \overset{*}{\gamma})^{\dagger d}{}_{d'},\\
&\text{(B)}\ \langle\overset{*}{\gamma}a'\overset{*}{\eta}(b'\overset{*}{\lambda}\overset{*}{\mu})\overset{*}{\nu}\,|\,\overset{*}{\gamma}c'\overset{*}{\lambda}\overset{*}{\kappa}(d'\overset{*}{\mu}\overset{*}{\nu})\rangle\\
&\qquad =A(\overset{*}{\gamma},\ \overset{*}{\eta}\overset{*}{\nu})^{a'}{}_{\overset{*}{a}}A(\overset{*}{\eta},\ \overset{*}{\lambda}\overset{*}{\mu})^{b'}{}_{\overset{*}{b}}\langle\gamma a\eta(b\lambda\mu)\nu\,|\,\gamma c\lambda\kappa(d\mu\nu)\rangle^{*} \qquad\qquad (5.2)\\
&\qquad\quad \times A(\overset{*}{\gamma},\ \overset{*}{\lambda}\overset{*}{\kappa})^{\dagger\overset{*}{c}}{}_{c'}A(\overset{*}{\kappa},\ \overset{*}{\mu}\overset{*}{\nu})^{\dagger\overset{*}{d}}{}_{d'},\\
&\text{(C)}\ \langle(\overset{*}{\lambda}\overset{*}{\mu})\!\uparrow a'\overset{*}{\eta},\ \overset{*}{\nu},\ \uparrow b'\overset{*}{\gamma}\,|\,\overset{*}{\lambda}(\overset{*}{\mu}\overset{*}{\nu})\!\uparrow c'\overset{*}{\kappa},\ \uparrow d'\overset{*}{\gamma}\rangle\\
&\qquad =A(\overset{*}{\lambda}\overset{*}{\mu}\!\uparrow,\ \overset{*}{\eta})^{a'}{}_{\overset{*}{a}}A(\overset{*}{\eta}\overset{*}{\nu}\!\uparrow,\ \overset{*}{\gamma})^{b'}{}_{\overset{*}{b}}\langle(\lambda\mu)\!\uparrow a\eta,\ \nu,\ \uparrow b\gamma\,|\,\lambda(\mu\nu)\!\uparrow c\kappa,\ \uparrow d\gamma\rangle^{*}\\
&\qquad\quad \times A(\overset{*}{\mu}\overset{*}{\nu}\!\uparrow,\ \overset{*}{\kappa})^{\dagger\overset{*}{c}}{}_{c'}A(\overset{*}{\lambda}\overset{*}{\kappa}\!\uparrow,\ \overset{*}{\gamma})^{\dagger\overset{*}{d}}{}_{d'}.
\end{aligned}$$

Note that the A matrices have been diagonalised with respect to the intermediate group labels. This factorisation is a consequence of the fact that this GH transformation can always be performed by a two-step transformation procedure, the one independent of the other (cf (3.21) and (3.26)).

The transposition symmetry generates two further symmetries. Their derivation may be seen more clearly by the schematic diagram of figure 1. This gives the relationship between the $2\times 3!$ possible ways of pairing L, M, N including permutations of the groups. The relationships are generated only by the associativity symmetry denoted by the full lines and the transposition symmetry denoted by the broken lines. Again we specify a process of forming irrep spaces in accordance with the way the

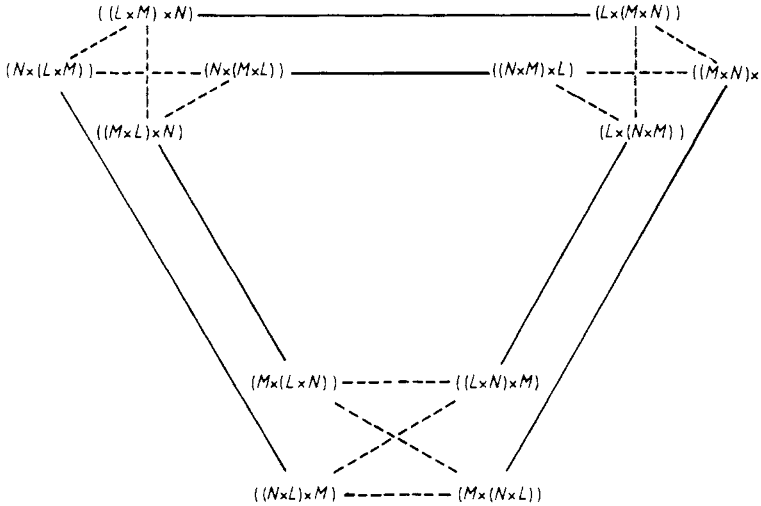

Figure 1.

direct product groups are taken. In traversing the various paths in figure 1, two independent identities are obtained which describe certain permutation symmetries of the associativity factors.

(1) The λ-ν transposition symmetry:

$$
\begin{aligned}
\text{(A)}\ &\langle \nu(\mu\lambda)a'\eta, b'\gamma|(\nu\mu)c'\kappa, \lambda, d'\gamma\rangle\\
&\quad = T(\lambda\mu, \eta)^{a'}{}_{a}T(\eta\nu, \gamma)^{b'}{}_{b}\langle(\lambda\mu)a\eta, \nu, b\gamma|\lambda(\mu\nu)c\kappa, d\gamma\rangle\\
&\qquad \times T(\mu\nu, \kappa)^{\dagger c}{}_{c'}T(\lambda\kappa, \gamma)^{\dagger d}{}_{d'},\\
\text{(B)}\ &\langle \gamma a'\nu\eta(b'\mu\lambda)|\gamma c'\kappa(d'\nu\mu)\lambda\rangle\\
&\quad = T(\gamma, \eta\nu)^{a'}{}_{a}T(\eta, \lambda\mu)^{b'}{}_{b}\langle\gamma a\eta(b\lambda\mu)\nu|\gamma c\lambda\kappa(d\mu\nu)\rangle\\
&\qquad \times T(\gamma, \lambda\kappa)^{\dagger c}{}_{c'}T(\kappa, \mu\nu)^{\dagger d}{}_{d'},\\
\text{(C)}\ &\langle \nu(\mu\lambda)\!\uparrow\! a'\eta, \uparrow\! b'\gamma|(\nu\mu)\!\uparrow\! c'\kappa, \lambda, \uparrow\! d'\gamma\rangle\\
&\quad = T(\lambda\mu\uparrow, \eta)^{a'}{}_{a}T(\eta\nu\uparrow, \gamma)^{b'}{}_{b}\langle(\lambda\mu)\!\uparrow\! a\eta, \nu, \uparrow\! b\gamma|\lambda(\mu\nu)\!\uparrow\! c\kappa, \uparrow\! d\gamma\rangle\\
&\qquad \times T(\mu\nu\uparrow, \kappa)^{\dagger c}{}_{c'}T(\lambda\kappa\uparrow, \gamma)^{\dagger d}{}_{d'}.
\end{aligned}
\tag{5.3}
$$

(2) The λ-μ-ν cyclic symmetry:

$$
\begin{aligned}
\text{(A)}\ &T(\eta\nu, \gamma)^{b}{}_{b_1}\langle(\lambda\mu)a\eta, \nu, b_1\gamma|\lambda(\mu\nu)c\kappa, d\gamma\rangle\\
&\qquad \times T(\kappa\lambda, \gamma)^{d}{}_{d_1}\langle(\mu\nu)c\kappa, \lambda, d_1\gamma|\mu(\nu\lambda)e\rho, f\gamma\rangle\\
&\qquad \times T(\rho\mu, \gamma)^{f}{}_{f_1}\langle(\nu\lambda)e\rho, \mu, f_1\gamma|\nu(\lambda\mu)a'\eta', b'\gamma\rangle\\
&\quad = \delta^{a}{}_{a'}\delta^{\eta}{}_{\eta'}\delta^{b}{}_{b'},\\
\text{(B)}\ &T(\gamma, \eta\nu)^{a}{}_{a_1}\langle\gamma a_1\eta(b\lambda\mu)\nu|\gamma c\lambda\kappa(d\mu\nu)\rangle\\
&\qquad \times T(\gamma, \kappa\lambda)^{c}{}_{c_1}\langle\gamma c_1\kappa(d\mu\nu)\lambda|\gamma e\mu\rho(f\nu\lambda)\rangle\\
&\qquad \times T(\gamma, \rho\mu)^{e}{}_{e_1}\langle\gamma e_1\rho(f\nu\lambda)\mu|\gamma a'\lambda\eta'(b'\mu\nu)\rangle\\
&\quad = \delta^{a}{}_{a'}\delta^{\eta}{}_{\eta'}\delta^{b}{}_{b'},
\end{aligned}
\tag{5.4}
$$

$$\begin{aligned}
\text{(C)}\quad & T(\eta\nu\uparrow,\gamma)^{b}{}_{b_1}\langle(\lambda\mu)\uparrow a\eta,\nu,\uparrow b_1\gamma|\lambda(\mu\nu)\uparrow c\kappa,\uparrow d\gamma\rangle\\
&\quad\times T(\kappa\lambda\uparrow,\gamma)^{d}{}_{d_1}\langle(\mu\nu)\uparrow c\kappa,\lambda,\uparrow d_1\gamma|\mu(\nu\lambda)\uparrow e\rho,\uparrow f\gamma\rangle\\
&\quad\times T(\rho\mu\uparrow,\gamma)^{f}{}_{f_1}\langle(\nu\lambda)\uparrow e\rho,\mu,\uparrow f_1\gamma|\nu(\lambda\mu)\uparrow a'\eta',\uparrow b'\gamma\rangle\\
&=\delta^{a}{}_{a'}\delta^{\eta}{}_{\eta'}\delta^{b}{}_{b'}.
\end{aligned}$$

From figure 1 note that although we have a threefold direct product group, the transposition symmetry involves only a twofold direct product. The corresponding T matrix can be chosen independent of the group labels of the third group. This factorisation has been used in obtaining the above identities.

Let us now consider GH chains containing a fourfold direct product group. As given earlier, there are five ways of forming the product by pairing but without permutations. These are given in figure 2. They are arranged in a particular order. Those connected by a line contain a common pairing. For example $(K\times L)$ is the common pairing of $(((K\times L)\times M)\times N)$ and $((K\times L)\times(M\times N))$, while $K\times K'$ with K' the resultant group of $L\times M\times N$ is the common pairing of $(K\times(L\times M)\times N)$ and $(K\times(L\times(M\times N)))$. Given the coupling, subduction or induction process by which the fourfold direct product space is paired, the transformation represented by the full line in figure 2 can be chosen independent of the common pairing. In this way the transformation coincides with the corresponding associativity factor. Furthermore, in traversing the path connected by the full lines we derive the following important identities for the various associativity factors:

$$\begin{aligned}
\text{(A)}\quad & \langle(\rho\mu)a\eta,\nu,b\gamma|\rho(\mu\nu)c\tau,d\gamma\rangle\langle(\kappa\lambda)c'\rho,\tau,d\gamma|\kappa(\lambda\tau)a'\varepsilon,b'\gamma\rangle\\
&=\langle(\kappa\lambda)c'\rho,\mu,a\eta|\kappa(\lambda\mu)d'\sigma,e\eta\rangle\langle(\kappa\sigma)e\eta,\nu,b\gamma|\kappa(\sigma\nu)f\varepsilon,b'\gamma\rangle\\
&\quad\times\langle(\lambda\mu)d'\sigma,\nu,f\varepsilon|\lambda(\mu\nu)c\tau,a'\varepsilon\rangle,\\
\text{(B)}\quad & \langle\gamma a\eta(b\rho\mu)\nu|\gamma c\rho\tau(d\mu\nu)\rangle\langle\gamma c\rho(d'\kappa\lambda)\tau|\gamma a'\kappa\varepsilon(b'\lambda\tau)\rangle\\
&=\langle\eta b\rho(d'\kappa\lambda)\mu|\eta e\kappa\sigma(c'\lambda\mu)\rangle\langle\gamma a\eta(e\kappa\sigma)\nu|\gamma a'\kappa\varepsilon(f\sigma\nu)\rangle\\
&\quad\times\langle\varepsilon f\sigma(c'\lambda\mu)\nu|\varepsilon b'\lambda\tau(d\mu\nu)\rangle,\\
\text{(C)}\quad & \langle(\rho\mu)\uparrow a\eta,\nu,\uparrow b\gamma|\rho(\mu\nu)\uparrow c\tau,\uparrow d\gamma\rangle\langle(\kappa\lambda)\uparrow c'\rho,\tau,\uparrow d\gamma|\kappa(\lambda\tau)\uparrow a'\varepsilon,\uparrow b'\gamma\rangle\\
&=\langle(\kappa\lambda)\uparrow c'\rho,\mu,\uparrow a\eta|\kappa(\lambda\mu)\uparrow d'\sigma,\uparrow e\eta\rangle\langle(\kappa\sigma)\uparrow e\eta,\nu,\uparrow b\gamma|\kappa(\sigma\nu)\uparrow f\varepsilon,\uparrow b'\gamma\rangle\\
&\quad\times\langle(\lambda\mu)\uparrow d'\sigma,\nu,\uparrow f\varepsilon|\lambda(\mu\nu)\uparrow c\tau,\uparrow a'\varepsilon\rangle.
\end{aligned}\tag{5.5}$$

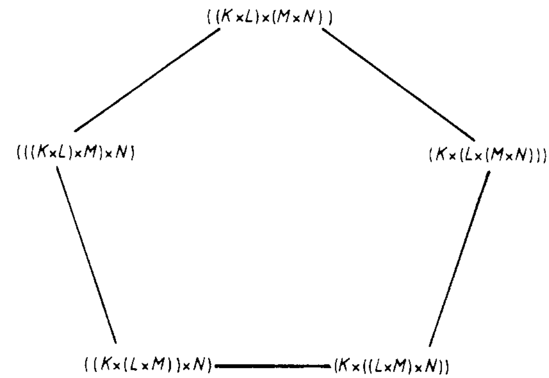

Figure 2.

The importance of these identities (5.5) and those giving the symmetries (5.2)-(5.4) lies in the use of the recursive method for calculating the values of these associativity factors. In particular if we consider the coupling process taking all groups to be equivalent to each other, that is $K \simeq L \simeq M = N \simeq \ldots \simeq G$, then the identities (5.2)-(5.5) are all well known in the Racah-Wigner coupling algebra. The recoupling factor is proportional to the higher symmetry $6j$ symbol (see Butler 1975, equation (9.13)). Substitution into (5.2)-(5.5) gives respectively the complex conjugation symmetry of the $6j$ symbol, the 'row-flip' symmetry of the $6j$ symbol, the Racah back-coupling relation and the Biedenharn-Elliott sum rule. These $6j$ relations have been used extensively in the building-up method for the calculation of $6j$ symbols for both the finite point groups (Butler and Wybourne 1976, Butler 1981, Butler and Reid 1979, Reid 1984) and some compact continuous groups (Butler *et al* 1978, 1979, Bickerstaff *et al* 1982). Algebraic formulae have also been obtained for some U_n $6j$ symbols (Haase and Butler 1985) and some symmetric group $6j$ symbols (Haase and Dirl 1985). A similar method of calculation employing the above equations could be applied to all three types of associativity factors. The procedure is as follows.

The phase freedom of each associativity factor must be determined. We have

$$
\begin{aligned}
&(\mathrm{A})\ \langle(\lambda\mu)\hat{a}\eta, \nu, \hat{b}\gamma|\lambda(\mu\nu)\hat{c}\kappa, \hat{d}\gamma\rangle \\
&\qquad = U(\lambda\mu, \eta)^{\hat{a}}{}_{a} U(\eta\nu, \gamma)^{\hat{b}}{}_{b} \langle(\lambda\mu)a\eta, \nu, b\gamma|\lambda(\mu\nu)c\kappa, d\gamma\rangle \\
&\qquad\quad \times U(\mu\nu, \kappa)^{+c}{}_{\hat{c}} U(\lambda\kappa, \gamma)^{+d}{}_{\hat{d}}, \\
&(\mathrm{B})\ \langle\gamma\hat{a}\eta(\hat{b}\lambda\mu)\nu|\gamma\hat{c}\lambda\kappa(\hat{d}\mu\nu)\rangle \\
&\qquad = U(\gamma, \eta\nu)^{\hat{a}}{}_{a} U(\eta, \lambda\mu)^{\hat{b}}{}_{b} \langle\gamma a\eta(b\lambda\mu)\nu|\gamma c\lambda\kappa(d\mu\nu)\rangle \\
&\qquad\quad \times U(\gamma, \lambda\kappa)^{+c}{}_{\hat{c}} U(\kappa, \mu\nu)^{+d}{}_{\hat{d}}, \\
&(\mathrm{C})\ \langle(\lambda\mu)\uparrow\hat{a}\eta, \nu, \uparrow\hat{b}\gamma|\lambda(\mu\nu)\hat{c}\kappa\uparrow, \hat{d}\gamma\rangle \\
&\qquad = T(\lambda\mu\uparrow, \eta)^{\hat{a}}{}_{a} U(\eta\nu\uparrow, \gamma)^{\hat{b}}{}_{b} \langle(\lambda\mu)\uparrow a\eta, \nu, \uparrow b\gamma|\lambda(\mu\nu)\uparrow c\kappa, \uparrow d\gamma\rangle \\
&\qquad\quad \times U(\mu\nu\uparrow, \kappa)^{+c}{}_{\hat{c}} U(\lambda\kappa\uparrow, \gamma)^{+d}{}_{\hat{d}}.
\end{aligned}
\tag{5.6}
$$

Note that each $U(\ldots)$ refers in general to a GH transformation for different GH chains. If the choices of the A and T matrices are to be used, the phase restrictions (3.36)-(3.37) and (4.14) must be incorporated into (5.6). If phase freedom exists for an associativity factor, an arbitrary choice can be made within the range specified by the unitary conditions and symmetry relations of the factor. If such a choice is to remain invariant under any further phase freedom choices, we have to impose a restriction on one phase freedom fixing it relative to the other three phase freedoms appearing in (5.6).

If no phase freedom exists for a factor, it may be calculated by either unitary conditions, the symmetry relations (5.3)-(5.4) or equation (5.5). In (5.5) we note that the five associativity factors belong in general to five different but related GH schemes. Thus the calculation would be performed by initially calculating associativity factors of four of the GH schemes which then completely fixes the relative phase freedoms in the fifth GH scheme. In this manner associativity factors are determined by building up from other associativity factors. In the case when all GH schemes are the same (that is, the Racah-Wigner coupling algebra) the so-called power of the irrep (see Butler and Wybourne 1976) becomes an important criterion by which irreps of one power are built up from irreps of lower power by coupling to the primitive (or defining) irrep of the group.

6. The defining factors

In Haase and Butler (1984a) we defined three transformation coefficients:

$$\begin{aligned}
&\text{(A) coupling coefficient} && \langle \mu\nu a\gamma i \,|\, \mu\nu mn\rangle, \\
&\text{(B) subduction coefficient} && \langle \gamma a\mu\nu mn \,|\, \gamma i\rangle, \\
&\text{(C) induction coefficient} && \langle \mu\nu\uparrow a\gamma i \,|\, \mu\nu\uparrow pmn\rangle.
\end{aligned} \tag{6.1}$$

Each coefficient defines a process by which one can form irreducible spaces from other irreducible spaces. Hence collectively we shall call them defining coefficients. By choosing an appropriate subgroup basis we obtained a corresponding defining factor:

$$\begin{aligned}
&\text{(A) coupling factor} && \langle \mu\nu a\gamma b\eta \,|\, (\mu c\lambda,\ \nu d\kappa)e\eta\rangle, \\
&\text{(B) subduction factor} && \langle \gamma a\mu(b\lambda)\nu(c\kappa) \,|\, \gamma a\eta e\lambda\kappa\rangle, \\
&\text{(C) induction factor} && \langle \mu\nu\uparrow a\gamma b\eta \,|\, q(\mu c\lambda,\ \nu d\kappa)\uparrow e\eta\rangle.
\end{aligned} \tag{6.2}$$

With regard to these factors we derive in this section three identities, one for each of the processes. These identities relate an associativity factor of some GH scheme with an associativity factor of a 'lower' GH scheme ('lower' in the sense of a reduction to a subgroup) and four defining factors—the type of factor involved depends on the process. In connection with these identities we discuss a method of calculating the various defining factors. The method is again based on the building-up method (Butler and Wybourne 1976). This is a systematic methodology for computing $3jm$ symbols (symmetrised coupling factors) using symmetry equations and the Wigner relation (see Butler 1981, equation (3.3.29)) which relates a $6j$ symbol of a group to that of a subgroup via four $3jm$ symbols. The corresponding symmetry and Wigner relations form the basis for the calculation of defining factors.

We begin with the symmetries. The complex conjugation symmetry can be obtained directly from (3.21) and (3.25):

$$\begin{aligned}
\text{(A)}\ &\langle \overset{*}{\mu}\overset{*}{\nu}a'\overset{*}{\gamma}b'\overset{*}{\eta} \,|\, (\overset{*}{\mu}c'\overset{*}{\lambda},\ \overset{*}{\nu}d'\overset{*}{k})e'\overset{*}{\eta}\rangle \\
&\quad = A(\overset{*}{\mu}\overset{*}{\nu},\ \overset{*}{\gamma})^{a'}{}_{\overset{*}{a}}A(\overset{*}{\gamma},\ \overset{*}{\eta})^{b'}{}_{\overset{*}{b}}\langle \mu\nu a\gamma b\eta \,|\, (\mu c\lambda,\ \nu d\kappa)e\eta\rangle^* \\
&\quad\quad \times A(\overset{*}{\mu},\ \overset{*}{\lambda})^{\dagger\overset{*}{c}}{}_{c'}A(\overset{*}{\nu},\ \overset{*}{\kappa})^{\dagger\overset{*}{d}}{}_{d'}A(\overset{*}{\lambda}\overset{*}{\kappa},\ \overset{*}{\eta})^{\dagger\overset{*}{e}}{}_{e'}, \\
\text{(B)}\ &\langle \overset{*}{\gamma}a'\overset{*}{\mu}(b'\overset{*}{\lambda})\overset{*}{\nu}(c'\overset{*}{\kappa}) \,|\, \overset{*}{\gamma}d'\overset{*}{\eta}e'\overset{*}{\lambda}\overset{*}{\kappa}\rangle \\
&\quad = A(\overset{*}{\gamma},\ \overset{*}{\mu}\overset{*}{\nu})^{a'}{}_{\overset{*}{a}}A(\overset{*}{\mu},\ \overset{*}{\lambda})^{b'}{}_{\overset{*}{b}}A(\overset{*}{\nu},\ \overset{*}{\kappa})^{c'}{}_{\overset{*}{c}}\langle \gamma a\mu(b\lambda)\nu(c\kappa) \,|\, \gamma d\eta e\lambda\kappa\rangle^* \\
&\quad\quad \times A(\overset{*}{\gamma},\ \overset{*}{\eta})^{\dagger\overset{*}{d}}{}_{d'}A(\overset{*}{\eta},\ \overset{*}{\lambda}\overset{*}{\kappa})^{\dagger\overset{*}{e}}{}_{e'}, \\
\text{(C)}\ &\langle \overset{*}{\mu}\overset{*}{\nu}\uparrow a'\overset{*}{\gamma}b'\overset{*}{\eta} \,|\, q(\overset{*}{\mu}c'\overset{*}{\lambda},\ \overset{*}{\nu}d'\overset{*}{\kappa})\uparrow e'\overset{*}{\eta}\rangle \\
&\quad = A(\overset{*}{\mu}\overset{*}{\nu}\uparrow,\ \overset{*}{\gamma})^{a'}{}_{\overset{*}{a}}A(\overset{*}{\gamma},\ \overset{*}{\eta})^{b'}{}_{\overset{*}{b}}\langle \mu\nu\uparrow a\gamma b\eta \,|\, q(\mu c\lambda,\ \nu d\kappa)\uparrow e\eta\rangle^* \\
&\quad\quad \times A(\overset{*}{\mu},\ \overset{*}{\lambda})^{\dagger\overset{*}{c}}{}_{c'}A(\overset{*}{\nu},\ \overset{*}{\kappa})^{\dagger\overset{*}{d}}{}_{d'}A(\overset{*}{\lambda}\overset{*}{\kappa}\uparrow,\ \overset{*}{\eta})^{\dagger\overset{*}{e}}{}_{e'},
\end{aligned} \tag{6.3}$$

while the transposition symmetry is a simple application of (4.10) and (4.11):

$$
\begin{aligned}
&\text{(A) } \langle \nu\mu a'\gamma b\eta \,|\, (\nu d\kappa, \mu c\lambda)e'\eta\rangle \\
&\qquad = T(\mu\nu,\gamma)^{a'}{}_{a}\langle \mu\nu a\gamma b\eta \,|\, (\mu c\lambda, \nu d\kappa)e\eta\rangle T(\lambda\kappa,\eta)^{\dagger e}{}_{e'}, \\
&\text{(B) } \langle \gamma a'\nu(c\kappa)\mu(b\lambda) \,|\, \gamma d\eta e'\kappa\lambda\rangle \\
&\qquad = T(\gamma,\mu\nu)^{a'}{}_{a}\langle \gamma a\mu(b\lambda)\nu(c\kappa) \,|\, \gamma d\eta e\lambda\kappa\rangle T(\eta,\lambda\kappa)^{\dagger e}{}_{e'}, \\
&\text{(C) } \langle \nu\mu\!\uparrow\! a'\gamma b\eta \,|\, q(\nu d\kappa, \mu c\lambda)\!\uparrow\! e'\eta\rangle \\
&\qquad = T(\mu\nu\!\uparrow,\gamma)^{a'}{}_{a}\langle \mu\nu\!\uparrow\! a\gamma b\eta \,|\, q(\mu c\lambda, \nu d\kappa)\!\uparrow\! e\eta\rangle T(\lambda\kappa\!\uparrow,\gamma)^{\dagger e}{}_{e'}.
\end{aligned}
\tag{6.4}
$$

Again suitable factorisations of A and T matrices have been employed.

The analogous Wigner relations can be obtained from the three equations (2.15), (2.9) and (5.9) of Haase and Butler (1984a). Detailed proofs are tedious to present and are not given here. In outline we take the following steps.

(i) Choose the appropriate 'lower' GH scheme.

(ii) Factorise the defining coefficients into defining factors and defining coefficients corresponding to the 'lower' GH scheme.

(iii) Recombine the 'lower' defining coefficients to form an associativity factor of the 'lower' GH scheme.

The results are as follows:

$$
\begin{aligned}
&\text{(A) } \langle (\lambda\mu)a\eta, \nu, b\gamma \,|\, \lambda(\mu\nu)c'\kappa, d'\gamma\rangle \\
&\qquad \times\langle \lambda\kappa d'\gamma e\rho \,|\, (\lambda a_1 v, \kappa b_1\tau)d\rho\rangle\langle \mu\nu c'\kappa b_1\tau \,|\, (\mu c_1\omega, \nu d_1\zeta)c\tau\rangle \\
&\quad = \langle \eta\nu b\gamma e\rho \,|\, (\eta e'\sigma, \nu d_1\zeta)b'\rho\rangle \\
&\qquad \times\langle \lambda\mu a\eta e'\sigma \,|\, (\lambda a_1 v, \mu c_1\omega)a'\sigma\rangle\langle (v\omega)a'\sigma, \zeta, b'\rho \,|\, v(\omega\zeta)c\tau, ,d\rho\rangle, \\
&\text{(B) } \langle \gamma a\eta(b\lambda\mu)\nu \,|\, \gamma c'\lambda\kappa(d'\mu\nu)\rangle \\
&\qquad \times\langle \kappa d'\mu(a_1\omega)\nu(b_1\zeta) \,|\, \kappa e_1\tau d\omega\zeta\rangle\langle \gamma c'\lambda(d_1 v)\kappa(c_1\tau) \,|\, \gamma e\rho c v\tau\rangle \\
&\quad = \langle \eta b\lambda(d_1 v)\mu(a_1\omega) \,|\, \eta e'\sigma b' v\omega\rangle \\
&\qquad \times\langle \gamma a\eta(e'\sigma)\nu(b_1\zeta) \,|\, \gamma e\rho a'\sigma\zeta\rangle\langle \rho a'\sigma(b' v\omega)\zeta \,|\, \rho c v\tau(d\omega\zeta)\rangle, \\
&\text{(C) } \langle (\lambda\mu)\!\uparrow\! a\eta, \nu, \uparrow\! b\gamma \,|\, \lambda(\mu\nu)\!\uparrow\! c'\kappa, \uparrow\! d'\gamma\rangle \\
&\qquad \times\langle \lambda\kappa\!\uparrow\! d'\gamma e\rho \,|\, q_1(\lambda a_1 v, \kappa b_1\tau)\!\uparrow\! d\rho\rangle\langle \mu\nu\!\uparrow\! c'\kappa b_1\tau \,|\, q_2(\mu c_1\omega, \nu d_1\zeta)\!\uparrow\! c\tau\rangle \\
&\quad = \langle \eta\nu\!\uparrow\! b\gamma e\rho \,|\, q_3(\eta e'\sigma, \nu d_1\zeta)\!\uparrow\! b'\rho\rangle \\
&\qquad \times\langle \lambda\mu\!\uparrow\! a\eta e'\sigma \,|\, q_4(\lambda a_1 v, \mu c_1\omega)\!\uparrow\! a'\sigma\rangle\langle (v\omega)\!\uparrow\! a'\sigma, \zeta, \uparrow\! b'\rho \,|\, v(\omega\zeta)\!\uparrow\! c\tau, \uparrow\! d\rho\rangle.
\end{aligned}
\tag{6.5}
$$

The calculation of the defining factors would be performed after the associativity factors for both the GH scheme and the 'lower' GH scheme have been obtained. One then proceeds by determining the phase freedom for each defining factor:

$$
\begin{aligned}
&\text{(A) } \langle \mu\nu\hat{a}\gamma\hat{b}\eta \,|\, (\mu\hat{c}\lambda, \nu\hat{d}\kappa)\hat{e}\eta\rangle \\
&\qquad = U(\mu\nu,\gamma)^{\hat{a}}{}_{a}U(\gamma,\eta)^{\hat{b}}{}_{b}\langle \mu\nu a\gamma b\eta \,|\, (\mu c\lambda, \nu d\kappa)e\eta\rangle \\
&\qquad\quad \times U(\mu,\lambda)^{\dagger c}{}_{\hat{c}}U(\nu,\kappa)^{\dagger d}{}_{\hat{d}}U(\lambda\kappa,\eta)^{\dagger e}{}_{\hat{e}},
\end{aligned}
$$

(B) $\langle \gamma\hat{a}\mu(\hat{b}\lambda)\nu(\hat{c}\kappa)|\gamma\hat{d}\eta\hat{e}\lambda\kappa\rangle$

$$= U(\gamma, \mu\nu)^{\hat{a}}{}_{a} U(\mu, \lambda)^{\hat{b}}{}_{b} U(\nu, \kappa)^{\hat{c}}{}_{c} \langle \gamma a\mu(b\lambda)\nu(c\kappa)|\gamma d\eta e\lambda\kappa\rangle \qquad (6.6)$$

$$\times U(\gamma, \eta)^{\dagger d}{}_{\hat{d}} U(\eta, \lambda\kappa)^{\dagger e}{}_{\hat{e}},$$

(C) $\langle \mu\nu\uparrow\hat{a}\gamma\hat{b}\eta|q(\mu\hat{c}\lambda, \nu\hat{d}\kappa)\uparrow\hat{e}\eta\rangle$

$$= U(\mu\nu\uparrow, \gamma)^{\hat{a}}{}_{a} U(\gamma, \eta)^{\hat{b}}{}_{b} \langle \mu\nu\uparrow a\gamma b\eta|q(\mu c\lambda, \nu d\kappa)\uparrow e\eta\rangle$$

$$\times U(\mu, \lambda)^{\dagger c}{}_{\hat{c}} U(\nu, \kappa)^{\dagger d}{}_{\hat{d}} U(\lambda\kappa\uparrow, \eta)^{\dagger e}{}_{\hat{e}}.$$

If a phase freedom exists for any factor, one makes a choice subject to the restrictions imposed by previous choices of A matrices, T matrices and associativity factors, and also the magnitude conditions given by unitarity. If no freedom remains for the defining factors, it is then calculated by either the unitarity condition, or (6.5) by choosing a suitable associativity factor. One important consideration which does not arise from any phase freedom arguments is the question of the 'orientation phase choice' (Reid and Butler 1982). This choice arose in the $3jm$ calculation of some point groups and $SU_3 \supset SO_3$, and appears to be related to the different possible embeddings of a subgroup in a group. However, as pointed out by Reid and Butler the exact nature of this choice is unclear, and the application of such choices in the present context of defining factors requires further investigation.

7. Summary

Our study of the Schur–Weyl duality which relates the symmetric and unitary groups has led to our formulating the GH transformation theory. In particular, the possible choices of the duality factors must be considered simultaneously with symmetries of the factors. The discussion in this paper of the global symmetries within GH transformation theory gives general results needed for the study of the particular symmetries of the duality factors.

We have also proposed in this paper a method for calculating the various associativity and defining factors. This method is a generalisation of the building-up method used to calculate $6j$ and $3jm$ symbols given by Butler and Wybourne (1976). That method has been successfully applied to point groups (Butler 1981) and some compact continuous groups (Butler *et al* 1978, 1979, Bickerstaff *et al* 1982). Algebraic formulae for the unitary and symmetric groups have also been obtained (Haase and Butler 1985, Haase and Dirl 1985). Numerical values and algebraic formulae of other symmetric and unitary group transformation factors are then possible, thus furthering the scope of the Schur–Weyl duality (cf Bickerstaff *et al* 1982, § 4 and Haase and Dirl 1985, § 7).

As a final remark the GH transformation theory does not only have application to the Schur–Weyl duality and its generalisation (Haase *et al* 1984). It is also relevant to the many-body system of particles which arise in many areas from molecular physics to atomic physics and nuclear physics to elementary particle physics. The techniques of the conventional Racah–Wigner algebra of angular momentum theory have long been used and are well known, but its generalisation (GH transformation theory) has been less often applied, with few attempts to use it in induced representation theory.

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